

# CATEGORICAL GREEN FUNCTORS ARISING FROM GROUP ACTIONS ON CATEGORIES

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**ABSTRACT.** In this paper we introduce the notion of a categorical Mackey functor. This categorical notion allows us to obtain new Mackey functors by passing to Quillen's  $K$ -theory of the corresponding abelian categories. In the case of an action by monoidal autoequivalences on a monoidal category the Mackey functor obtained at the level of Grothendieck rings has in fact a Green functor structure.

## 1. INTRODUCTION AND MAIN RESULTS

A Mackey functor or (a  $G$ -functor) is a family  $\{a(K)\}_{K \leq G}$  of abelian groups equipped with three types of maps: induction, conjugation, and restriction, satisfying some certain compatibility axioms, see for example [6]. Typical examples, include among others, the cohomology groups  $\{H^n(K, M)\}_{K \leq G}$  and the character rings  $\{R(K)\}_{K \leq G}$ .

In [1, 8, 11] it is shown that for any group  $G$  of automorphisms of a number field  $k$ , the class group of the ring of integers of the fixed field  $\{k^H\}_{H \leq G}$  is a  $G$ -functor. These results were extended in [7] by showing that  $\{K_i(S^H)\}_{H \leq G}$  is a  $G$ -functor, whenever  $R \subseteq S$  is a Galois extension of commutative rings with Galois group  $G$ .

The main goal of this paper is to construct a categorical version of a Mackey functor. The main source of examples for such categorical functors is given by the group actions on categories. It is shown that group actions on abelian category give rise to Mackey functors while monoidal group actions on monoidal categories give rise to categorical Green functors. By passing to the  $K$ -theory a categorical Mackey functor give rise to a classical Mackey functor. In this way we give new examples of Mackey and Green functors generalizing the examples given [7, 2].

Let  $G$  be a finite group acting on the abelian category  $\mathcal{C}$ . For any subgroup  $H$  of  $G$  the left adjoint functor of the forgetful functor  $\text{Res}_H^G : \mathcal{C}^G \rightarrow \mathcal{C}^H$  was recently described in [3]. This functor is denoted by  $\text{Ind}_H^G : \mathcal{C}^H \rightarrow \mathcal{C}^G$  and can be regarded as a generalization of the induction functor from  $\text{Rep}(H)$  to  $\text{Rep}(G)$ .

Using this notion of induction and restriction we introduce the concept of categorical Mackey and Green functors that category the classical concepts. Note that the induction and restriction functors were also considered in various other contexts as categorized constructions of representation theory, see for example [12].

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Our first main result is the following:

**Theorem 1.1.** *Let  $G$  be a finite group acting on the abelian category  $\mathcal{C}$  by  $T : G \rightarrow \underline{\text{Aut}}(\mathcal{C})$ .*

- (1) *Then the functor  $H \mapsto \mathcal{C}^H$  defines a categorical Mackey functor over  $\text{Vec}_k$ .*
- (2) *Moreover if  $\mathcal{C}$  is a  $k$ -linear monoidal category and the action of  $G$  is by monoidal autoequivalences then the above functor is a categorical Green functor over  $\mathcal{C}^G$ .*

The proof of the above results uses a Mackey type decomposition for the above induced functor when restricted to various subgroups:

**Theorem 1.2.** *Suppose that a finite group  $G$  acts on the abelian category  $\mathcal{C}$  via  $T : G \rightarrow \underline{\text{Aut}}(\mathcal{C})$ . Let  $K$  and  $L$  be any two subgroups of a subgroup  $H \leq G$  and  $M \in \mathcal{C}^H$ .*

1) *Then*

$$(1.3) \quad \text{Res}_K^H(\text{Ind}_L^H(M)) \simeq \bigoplus_{x \in K \backslash H/L} \text{Ind}_{K \cap {}^x L}^K(\text{Res}_{{}^x L \cap K}^{{}^x L}({}^x c_{L,x}(M)))$$

where  ${}^x L := xLx^{-1}$  and the equivariant structure for  $c_{L,x}(M) \in \mathcal{C}^{{}^x L}$  is given as in Lemma 4.1.

2) *If  $\mathcal{C}$  is a  $k$ -linear monoidal category and the action of  $G$  on  $\mathcal{C}$  is by monoidal autoequivalences then the above isomorphism is of  $\mathcal{C}^G$ -module functors.*

As an application of Theorem 4.25 we obtain the following corollary:

**Corollary 1.4.** *Let  $G$  be a finite group acting on the abelian category  $\mathcal{C}$  by  $T : G \rightarrow \underline{\text{Aut}}(\mathcal{C})$ .*

- (1) *Then for all  $i \geq 0$  the functor  $H \mapsto K_i(\mathcal{C}^H)$  defines a  $G$ -functor  $M_i$  with the following structure maps:*
  - (a) *Restriction  $R_K^H : K_i(\mathcal{C}^H) \rightarrow K_i(\mathcal{C}^K)$  is the map induced by the forgetful functor  $\text{Res}_K^H : \mathcal{C}^H \rightarrow \mathcal{C}^K$ ,*
  - (b) *Induction  $I_K^H : K_i(\mathcal{C}^H) \rightarrow K_i(\mathcal{C}^K)$  is the map induced by the induction functor  $\text{Ind}_K^H : \mathcal{C}^K \rightarrow \mathcal{C}^H$ ,*
  - (c) *Conjugation  $c_{H,x} : K_i(\mathcal{C}^H) \rightarrow K_i(\mathcal{C}^{{}^x H})$  is the map induced by the functor  $T^x : \mathcal{C}^H \rightarrow \mathcal{C}^{{}^x H}$ .*
- (2) *If  $\mathcal{C}$  is a  $k$ -linear monoidal category over  $k$  and  $G$  is a finite group acting on  $\mathcal{C}$  by monoidal autoequivalences then  $H \mapsto K_0(\mathcal{C}^H)$  defines a Green functor on  $G$  over  $k$ .*

Shortly, this paper is organized as follows. In Section 2 we recall some basic results on abelian categories and group actions on them. The construction of the adjoint functor  $\text{Ind}_H^G$  mentioned above is also recalled in this section.

In Section 3 we first recall the definition of the classical Mackey and Green functors. Then we present the new categorical notions of Mackey and Green functors.

Section 4 is devoted to the proof of Theorem 1.1 and Theorem 1.2. In Theorem 4.25 we show that by passing to Quillen's K-theory a categorical Mackey functor gives rise to a classical Mackey functors. (see Theorem 4.25). This proves Corollary 1.4.

## 2. GROUP ACTIONS ON $k$ -LINEAR CATEGORIES

Fix a commutative ring  $k$ . Recall that a  $k$ -linear category is an abelian category in which the hom-sets are  $k$ -vector spaces and the compositions of morphisms are  $k$ -bilinear. A  $k$ -linear functor between  $k$ -linear categories is a functor which is linear on all hom-spaces. Recall that an adjunction between categories  $\mathcal{C}$  and  $\mathcal{D}$  is a pair of functors,  $F : \mathcal{D} \rightarrow \mathcal{C}$  and  $G : \mathcal{C} \rightarrow \mathcal{D}$  and a family of bijections  $\text{hom}_{\mathcal{C}}(FY, X) \cong \text{hom}_{\mathcal{D}}(Y, GX)$

which is natural in the variables  $X$  and  $Y$ . The functor  $F$  is called a left adjoint functor, while  $G$  is called a right adjoint functor. The relationship  $F$  is left adjoint to  $G$  (or equivalently,  $G$  is right adjoint to  $F$ ) is sometimes written  $F \dashv G$ .

A *monoidal category* is a category  $\mathcal{C}$  equipped with a bifunctor  $\otimes: \mathcal{C} \times \mathcal{C} \rightarrow \mathcal{C}$  called the *monoidal product*, and an object  $\mathbf{1}_{\mathcal{C}}$  called the *unit object*. The category  $\mathcal{C}$  has a natural isomorphism  $\alpha$ , called *associativity constraint*, given by  $\alpha_{A,B,C}: (A \otimes B) \otimes C \xrightarrow{\sim} A \otimes (B \otimes C)$  for all  $A, B, C \in \mathcal{C}$ . There are also two natural isomorphisms  $l_A$  and  $r_A$ , respectively called left and respectively *right unitor*, with components  $l_A: I \otimes A \cong A$  and  $r_A: A \otimes I \cong A$ . These natural transformations satisfy some coherence conditions expressed by the fact that pentagon and the triangle diagram commute, see e.g. [5].

Recall that a *unitary monoidal functor*  $F: \mathcal{C} \rightarrow \mathcal{D}$  between two monoidal categories is a  $k$ -linear functor  $F$  together with a natural transformation  $F_2: F(- \otimes -) \rightarrow F(-) \otimes F(-)$  and a unit isomorphism  $F_0: F(\mathbf{1}_{\mathcal{C}}) \rightarrow \mathbf{1}_{\mathcal{D}}$  satisfying the compatibility of the following hexagon and unit axioms (see for example [5]).

$$\begin{array}{ccc}
 (F(A) \otimes F(B)) \otimes F(C) & \xrightarrow{\alpha_{F(A), F(B), F(C)}} & F(A) \otimes (F(B) \otimes F(C)) \\
 \downarrow F_2^{A,B} \otimes 1 & & \downarrow 1 \otimes F_2^{B,C} \\
 F(A \otimes B) \otimes F(C) & & F(A) \otimes F(B \otimes C) \\
 \downarrow F_2^{A \otimes B, C} & & \downarrow F_2^{A, B \otimes C} \\
 F((A \otimes B) \otimes C) & \xrightarrow{F(\alpha_{A,B,C})} & F(A \otimes (B \otimes C)).
 \end{array}$$

(Diagram H)

$$\begin{array}{ccc}
 F(\mathbf{1}_{\mathcal{C}} \otimes A) & \xrightarrow{F(l_A)} & F(A) \\
 \downarrow F_2^{\mathbf{1}_{\mathcal{C}}, A} & & \uparrow l_{F(A)} \\
 F(\mathbf{1}_{\mathcal{C}}) \otimes F(A) & \xrightarrow{F_0 \otimes 1} & \mathbf{1}_{\mathcal{D}} \otimes F(A)
 \end{array}$$

(Diagram LU)

$$\begin{array}{ccc}
 F(A \otimes \mathbf{1}_{\mathcal{C}}) & \xrightarrow{F(r_A)} & F(A) \\
 \downarrow F_2^{A, \mathbf{1}_{\mathcal{C}}} & & \uparrow r_{F(A)} \\
 F(A) \otimes F(\mathbf{1}_{\mathcal{C}}) & \xrightarrow{1 \otimes F_0} & F(A) \otimes \mathbf{1}_{\mathcal{D}}
 \end{array}$$

(Diagram RU)

In particular the naturality of  $F_2$  with respect to the morphisms can be written as

$$(2.1) \quad (F(u) \otimes F(v)) F_2^{M,N} = F_2^{M',N'} F(u \otimes v)$$

for all morphisms  $M \xrightarrow{u} M'$  and  $N \xrightarrow{v} N'$  in  $\mathcal{C}$ . Composition of two monoidal functors  $\mathcal{C} \xrightarrow{G} \mathcal{D} \xrightarrow{F} \mathcal{E}$  is also a monoidal functor with

$$(2.2) \quad (F \circ G)_2^{M,N} := F_2^{G(M), G(N)} \circ F(G_2^{M,N})$$

A *natural monoidal transformation*  $\tau : F \rightarrow G$  between two monoidal functors is a natural transformation satisfying the following compatibility condition:

$$(2.3) \quad G_2^{M,N} \tau_{M \otimes N} = (\tau_M \otimes \tau_N) F_2^{M,N}$$

for any objects  $M, N \in \mathcal{C}$ .

If  $\mathcal{C}$  is a monoidal category, a *left module category* over  $\mathcal{C}$  is a category  $\mathcal{M}$  endowed with an *action* functor  $\boxtimes : \mathcal{C} \times \mathcal{M} \rightarrow \mathcal{M}$  with *module associativity constraints*  $a_{A,B,M} : A \boxtimes (B \boxtimes M) \xrightarrow{\sim} (A \otimes B) \boxtimes M$  and a unit constraint  $u_M : \mathbf{1} \boxtimes M \xrightarrow{\sim} M$  satisfying a pentagon and a triangle coherence axiom, see [5]. A *module functor*  $F : \mathcal{M} \rightarrow \mathcal{N}$  between two  $\mathcal{C}$ -module categories  $\mathcal{M}, \mathcal{N}$  is a functor with an additional module structure such that  $(F_2)^{A,M} : F(A \boxtimes M) \simeq A \boxtimes F(M)$  satisfying the following diagrams:

$$\begin{array}{ccc} F(\mathbf{1} \boxtimes M) & \xrightarrow{F(u_M)} & F(M) \\ & \searrow F_2^{\mathbf{1},M} & \nearrow u_{F(M)} \\ & \mathbf{1} \boxtimes F(M) & \end{array}$$

(Diagram UM)

and

$$\begin{array}{ccc} F((A \otimes B) \boxtimes M) & \xrightarrow{F_2^{A \otimes B, M}} & (A \otimes B) \boxtimes F(M) \\ \downarrow F(\alpha_{A,B,F(M)}) & & \downarrow \alpha_{A,B,F(M)} \\ F(A \boxtimes (B \boxtimes M)) & & A \boxtimes (B \boxtimes F(M)) \\ & \searrow F_2^{A,B \boxtimes M} & \nearrow 1 \otimes F_2^{B,M} \\ & A \boxtimes F(B \otimes M) & \end{array}$$

(Diagram HM)

Let  $\mathcal{C}$  be a monoidal category and  $\mathcal{M}, \mathcal{N}$  be two left  $\mathcal{C}$ -module categories. Let also  $F, G : \mathcal{M} \rightarrow \mathcal{N}$  be two  $\mathcal{C}$ -module functors. A morphism between  $F$  and  $G$  is a natural transformation  $T : F \rightarrow G$  such that the following diagram commutes

$$\begin{array}{ccc} F_1(P \otimes M) & \xrightarrow{T_{P \otimes M}} & F_2(P \otimes M) \\ \downarrow F_2^{P,M} & & \downarrow G_2^{P,M} \\ P \otimes F_1(M) & \xrightarrow{1_P \otimes T_M} & P \otimes F_2(M) \end{array}$$

for any  $P \in \mathcal{C}$  and any  $M \in \mathcal{M}$ . Let  $R, S : \mathcal{C} \rightarrow \mathcal{D}$  be two functors and  $N : R \rightarrow S$  be a natural transformation between  $R$  and  $S$ . For any functor  $F : \mathcal{D} \rightarrow \mathcal{E}$  one can define the natural transformation  $FN : FR \rightarrow FS$  by  $(FN)_X := F(N_X) : FR(X) \rightarrow FS(X)$ .

Moreover for any functor  $G : \mathcal{E} \rightarrow \mathcal{C}$  one can also define  $N_G : RG \rightarrow SG$  as natural transformation by  $(N_G)_X := N_{G(X)}$ .

**2.1. Group actions on abelian categories.** Let  $\mathcal{C}$  be an abelian category. Denote by  $\underline{\text{Aut}}(\mathcal{C})$  the category whose objects are *exact* autoequivalences of  $\mathcal{C}$  and morphisms are natural transformations between them. Then  $\underline{\text{Aut}}(\mathcal{C})$  is a monoidal category where the tensor product is defined as the composition of autoequivalences. For a finite group  $G$  let  $\text{Cat}(G)$  denote the monoidal category whose objects are elements of  $G$ , the only morphisms are the identities, and the tensor product is given by multiplication in  $G$ . An action of a finite group  $G$  on  $\mathcal{C}$  consists of a unitary monoidal functor  $T : \text{Cat}(G) \rightarrow \underline{\text{Aut}}(\mathcal{C})$ . Thus, for every  $g \in G$ , we have a functor  $T^g : \mathcal{C} \rightarrow \mathcal{C}$  and a collection of natural isomorphisms

$$T_2^{g,h} : T^g T^h \rightarrow T^{gh}, \quad g, h \in G,$$

which give the monoidal structure of  $T$ . The monoidal unit of  $T$  is denoted by  $T_0 : \text{id}_{\mathcal{C}} \rightarrow T^1$  where  $1 \in G$  is the unit of the group  $G$ . By the definition of the monoidal functor, the monoidal structure  $T_2$  satisfies the following conditions:

$$(2.4) \quad (T_2^{g,h,l})_M (T_2^{g,h})_{T^l(M)} = (T_2^{g,hl})_M T^g((T_2^{h,l})_M),$$

$$(2.5) \quad (T_2^{g,1})_M T^g(T_{0M}) = (T_2^{1,g})_M (T_0)_{T^g(M)} = \text{id}_{T^g(M)},$$

for all objects  $M \in \mathcal{C}$ , and for all  $g, h, l \in G$ . See [4, Subsection 4.1]. Note that by the naturality of  $T_2^{g,h}$ ,  $g, h \in G$ , can be written as

$$(2.6) \quad T^{gh}(f) (T_2^{g,h})_N = (T_2^{g,h})_M T^g T^h(f),$$

for every morphism  $f : M \rightarrow N$  in  $\mathcal{C}$ . We shall assume in what follows that  $T^1 = \text{id}_{\mathcal{C}}$  and  $T_0, T_2^{g,1}, T_2^{1,g}$  are also identities. We say that  $G$  acts *k-linearly* on the *k-linear* category  $\mathcal{C}$  if  $T^g$  is a *k-linear* autoequivalence for any  $g \in G$ .

**Example 2.7.** Suppose that  $G$  acts by ring automorphisms on a *k-algebra*  $S$ . Then  $G$  acts on  $S\text{-mod}$  via the following action:  $T^g(M) = M$  as abelian groups and the  $S$ -action on  $T^g(M)$  is given by  $s \cdot {}^g m := (g^{-1} \cdot s)m$ . In this case one can take  $(T_2^{g,h})_M = \text{id}_M$  for all  $g, h \in G$ .

**2.2. On the equivariantized category.** Suppose that  $G$  acts on the abelian category  $\mathcal{C}$ . Let  $\mathcal{C}^G$  denote the corresponding *equivariantized* category. Recall that  $\mathcal{C}^G$  is an abelian category whose objects are  $G$ -equivariant objects of  $\mathcal{C}$ . They consist of pairs  $(M, \mu)$ , where  $M$  is an object of  $\mathcal{C}$  and  $\mu = (\mu_M^g)_{g \in G}$  is a collection of isomorphisms  $\mu_M^g : T^g(M) \rightarrow M$  in  $\mathcal{C}$  satisfying the following:

$$(2.8) \quad \mu_M^g T^g(\mu_M^h) = \mu_M^{gh} (T_2^{g,h})_M, \quad \forall g, h \in G, \quad \mu_M^1 T_{0M} = \text{id}_M.$$

We say that an object  $M$  of  $\mathcal{C}$  is *G-equivariant* if there exists such a collection  $\mu = (\mu^g)_{g \in G}$  so that  $(M, \mu) \in \mathcal{C}^G$ . Note that the equivariant structure  $\mu$  is not necessarily unique.

A morphism  $f : (M, \mu_M) \rightarrow (N, \mu_N)$  in  $\mathcal{C}^G$  is a morphism in  $f : M \rightarrow N$  such that

$$(2.9) \quad \mu_N^g T^g(f) = f \mu_M^g.$$

**Example 2.10.** It is easy to verify that in the case of the previous example one has that  $(S\text{-mod})^G \simeq S \# kG\text{-mod}$ , the category of  $S \# kG$ -modules.

**2.3. Induction functors as left adjoints of restriction functors.** Suppose that a finite group  $G$  acts on the abelian category  $\mathcal{C}$  and let  $H \leq G$  be a subgroup. Let  $\mathcal{R}$  be a set of representative elements for the left cosets  $\{Hx \mid x \in G\}$  of  $H$  in  $G$ . Thus one can write  $G$  as a disjoint union  $G = \cup_{t \in \mathcal{R}} tH$ . Set, for all  $(V, \mu) \in \mathcal{C}^H$ ,

$$(2.11) \quad \text{Ind}_H^G(V, \mu) := (\oplus_{t \in \mathcal{R}} T^t(V), \nu) \in \mathcal{C}^G$$

where for all  $g \in G$  the equivariant structure of  $\nu_{\text{Ind}_H^G(V, \mu)}^g : \oplus_{t \in \mathcal{R}} T^g T^t(V) \rightarrow \oplus_{t \in \mathcal{R}} T^t(V)$  is defined componentwise by the formula

$$(2.12) \quad \nu^{g,t} = T^s(\nu^h)(T_2^{s,h})^{-1} T_2^{g,t} : T^g T^t(V) \rightarrow T^s(V).$$

Here the elements  $h \in H$  and  $s \in \mathcal{R}$  are uniquely determined by the relation  $gt = sh$ .

**Proposition 2.13.** *Suppose that the finite group  $G$  acts on the  $k$ -linear category  $\mathcal{C}$ . With the above notations one has that  $\text{Ind}_H^G : \mathcal{C}^H \rightarrow \mathcal{C}^G$  is a  $k$ -linear left adjoint of the restriction functor  $\text{Res}_H^G : \mathcal{C}^G \rightarrow \mathcal{C}^H$ .*

*Proof.* The proof of [3, Proposition 2.9] works verbatim in the case of an abelian category  $\mathcal{C}$ .  $\square$

Note that the functor  $\text{Ind}_H^G$  as defined above, depends on the set of representative elements  $\mathcal{R}$ . Since the adjoint of a functor is unique up to isomorphism it follows that for a different set of representative elements one obtains an isomorphic functor.

**2.4. Action by monoidal autoequivalences.** Suppose that  $\mathcal{C}$  is a  $k$ -linear monoidal category and consider  $\underline{\text{Aut}}_{\otimes}(\mathcal{C})$  the full subcategory of  $\underline{\text{Aut}}(\mathcal{C})$  consisting of  $k$ -linear monoidal autoequivalences of  $\mathcal{C}$ .

We say that  $G$  acts on  $\mathcal{C}$  by *monoidal autoequivalences*, if there is a unitary monoidal functor  $T : \underline{G} \rightarrow \underline{\text{Aut}}_{\otimes} \mathcal{C}$ . Thus for any  $g \in G$  there is,  $T^g : \mathcal{C} \rightarrow \mathcal{C}$  a monoidal autoequivalence of  $\mathcal{C}$  satisfying the conditions from above. Moreover,  $T^g$  is endowed with a monoidal structure  $(T_2^g)^{M,N} : T^g(M \otimes N) \rightarrow T^g(M) \otimes T^g(N)$ , for all  $M, N \in \mathcal{C}$  and  $T_2^{g,h} : T^g T^h \rightarrow T^{gh}$  are natural isomorphisms of monoidal functors, for all  $g, h \in G$ . Thus, for all  $g, h \in G$  and  $M, N \in \mathcal{C}$ , Equation (2.3) becomes:

$$(2.14) \quad (T_2^{gh})^{M,N} (T_2^{g,h})_{M \otimes N} = ((T_2^{g,h})_M \otimes (T_2^{g,h})_N) (T_2^g)^{T^h(M), T^h(N)} T^g((T_2^h)_{M,N}).$$

### 3. CATEGORICAL MACKEY AND GREEN FUNCTORS

In this section we introduce the notion of categorical Mackey and Green functors and we prove some of their properties. First we recall the notion of Mackey and Green functors over rings.

**3.1. Classical Mackey functors.** Let  $G$  be a finite group. A *Mackey functor* (or a  $G$ -functor) over a ring  $R$  is a collection of  $R$ -modules  $\{a(H)\}_{H \leq G}$  together with morphisms  $I_K^H : a(K) \rightarrow a(H)$ ,  $R_K^H : a(H) \rightarrow a(K)$  and  $c_{H,g} : a(H) \rightarrow a({}^g H)$  for all subgroups  $H$  and  $K$  of  $G$  with  $K \leq H$  and for all  $g \in G$ . This datum satisfies the following compatibility conditions:

- (M0)  $I_H^H, R_H^H, c_{H,h} : M(H) \rightarrow M(H)$  are the identity morphisms for all subgroups  $H$  and  $h \in H$ .
- (M1)  $R_J^K R_K^H = R_J^H$  for all subgroups  $J \leq K \leq H$ .
- (M2)  $I_K^H I_J^K = I_J^H$ , for all subgroups  $J \leq K \leq H$ .
- (M3)  $c_{H,g} c_{{}^g H, h} = c_{H,gh}$ , for all  $H \leq G$  and  $g, h \in G$ .
- (M4) For any subgroups  $K, H \leq G$  the following Mackey relation is satisfied:

$$(3.1) \quad R_H^G I_K^G = \bigoplus_{x \in H \backslash G / K} I_{xK \cap H}^H R_{xK \cap H}^{xK} c_{K,x}.$$

Moreover, a *Green functor* over a commutative ring  $R$ , is a  $G$ -functor  $a$  such that for any subgroup  $H$  of  $G$  one has that  $a(H)$  is an associative  $R$ -algebra with identity and satisfying the following:

- (G1)  $R_K^H$  and  $c_{H,g}$  are always unitary  $R$ -algebra homomorphisms,
- (G2)  $I_K^H(aR_K^H(b)) = I_K^H(a)b$ ,
- (G3)  $I_K^H(R_K^H(b)a) = bI_K^H(a)$  for all subgroups  $K \leq H$  and all  $a \in a(K)$  and  $b \in a(H)$ .

**Remark 3.2.** Note that the compatibility conditions (G2) (and respectively (G3)) can be expressed as the fact that the induction maps  $I_K^H$  are morphisms of right (and respectively left)  $a(H)$ -modules.

**3.2. Categorical Mackey functors.** Let  $\mathcal{S}$  be a given  $k$ -linear monoidal category. A *categorical Mackey functor* (or a categorical  $G$ -functor) over  $\mathcal{S}$  is a collection of

- (1)  $\mathcal{S}$ -module categories  $\{\mathcal{M}(L)\}_{L \leq G}$
- (2)  $\mathcal{S}$ -module functors  $\mathcal{I}_K^H : \mathcal{M}(K) \rightarrow \mathcal{M}(H)$ ,
- (3)  $\mathcal{S}$ -module functors  $\mathcal{R}_K^H : \mathcal{M}(H) \rightarrow \mathcal{M}(K)$  and
- (4)  $\mathcal{S}$ -module functors  $c_{L,g} : \mathcal{M}(L) \rightarrow \mathcal{M}(^g L)$

for all subgroups  $K \leq H$  of  $G$  and for all  $g \in G$ . Moreover, the following compatibilities conditions are satisfied:

- (CM0)  $\mathcal{I}_H^H, \mathcal{R}_H^H, c_{H,h} : \mathcal{M}(H) \rightarrow \mathcal{M}(H)$  are  $k$ -linear isomorphic to the identity morphisms, for any  $h \in H$ .
- (CM1) There are natural transformations which are isomorphisms of module functors

$$\mathbf{R}_{J,H}^K : \mathcal{R}_J^H \xrightarrow{\cong} \mathcal{R}_J^K \mathcal{R}_K^H$$

for all  $J \leq K \leq H$

- (CM2) There are natural transformations which are isomorphisms of module functors

$$\mathbf{I}_{J,H}^K : \mathcal{I}_J^H \xrightarrow{\cong} \mathcal{I}_K^H \mathcal{I}_J^K$$

for all  $J \leq K \leq H$ .

- (CM3) There are natural transformations which are isomorphisms of module functors

$$\mathbf{C}_{a,b}^H : c_{H,ab} \xrightarrow{\cong} c_{^b H,a} c_{H,b},$$

for all  $H \leq G$  and  $a, b \in G$ .

- (CM4) For any subgroups  $K \leq H$  there are isomorphisms of module functors

$$\mathbf{C}\mathbf{I}_{H,a}^K : c_{H,a} \mathcal{I}_K^H \xrightarrow{\cong} \mathcal{I}_a^H c_{K,a}$$

- (CM5) For any subgroups  $K \leq H$  there are natural transformation which are isomorphisms of module functors

$$\mathbf{C}\mathbf{R}_{H,a}^K : c_{K,a} \mathcal{R}_K^H \xrightarrow{\cong} \mathcal{R}_a^H c_{H,a}$$

- (CM6) For any two subgroups  $L, K \leq H$  one has an isomorphism

$$\mathbf{R}\mathbf{I}_{L,K}^H : \mathcal{R}_L^H \circ \mathcal{I}_K^H \xrightarrow{\cong} \bigoplus_{x \in L \setminus H/K} \mathcal{I}_{xK \cap L}^L \circ \mathcal{R}_{xK \cap L}^{xK} \circ c_{K,x}.$$

as  $\mathcal{S}$ -module functors.

Moreover for any tower  $J \leq K \leq L \leq H$  of subgroups of  $G$  and any  $a, b, c \in G$  one has that we have the following coherence relations between these natural transformations:

$$(3.3) \quad \mathbf{I}_{L,H}^H = \text{id}_{\mathcal{I}_L^H} \quad \text{and} \quad \mathbf{I}_{L,H}^L = \text{id}_{\mathcal{I}_L^H}$$

$$(3.4) \quad \mathbf{R}_{L,H}^H = \text{id}_{\mathcal{R}_L^H} \quad \text{and} \quad \mathbf{R}_{L,H}^L = \text{id}_{\mathcal{R}_L^H}$$

$$(3.5) \quad \mathbf{C}_{a,1}^H = \text{id}_{c_{H,a}} = \mathbf{C}_{1,a}^H$$

$$\begin{array}{ccc}
 \mathcal{R}_J^K \mathcal{R}_K^L \mathcal{R}_L^H & \xleftarrow{\mathcal{R}_J^K (\mathbf{R}_{K,H}^L)} & \mathcal{R}_J^K \mathcal{R}_K^H \\
 \uparrow (\mathbf{R}_{J,L}^K)_{\mathcal{R}_L^H} & & \uparrow \mathbf{R}_{J,H}^K \\
 \mathcal{R}_J^L \mathcal{R}_L^H & \xleftarrow{\mathbf{R}_{J,H}^L} & \mathcal{R}_J^H
 \end{array}$$

(Diagram R)

$$\begin{array}{ccc}
 \mathcal{I}_L^H \mathcal{I}_K^L \mathcal{I}_J^K & \xleftarrow{(\mathbf{I}_{K,H}^L)_{\mathcal{I}_J^K}} & \mathcal{I}_K^H \mathcal{I}_J^K \\
 \uparrow \mathcal{I}_L^H (\mathbf{I}_{J,L}^K) & & \uparrow \mathbf{I}_{J,H}^K \\
 \mathcal{I}_L^H \mathcal{I}_J^L & \xleftarrow{\mathbf{I}_{J,H}^L} & \mathcal{I}_J^H
 \end{array}$$

(Diagram I)

$$\begin{array}{ccc}
 c_{H,abc} & \xrightarrow{\mathbf{C}_{ab,c}^H} & c \ c_{H,ab} c_{H,c} \\
 \downarrow \mathbf{C}_{a,bc}^H & & \downarrow (\mathbf{C}_{a,b}^H)_{c_{H,c}} \\
 c \ bc_{H,a} c_{H,bc} & \xrightarrow{c \ bc_{H,a} (\mathbf{C}_{b,c}^H)} & c \ bc_{H,a} c \ c_{H,b} c_{H,c}
 \end{array}$$

(Diagram C)

$$\begin{array}{ccc}
 c_{K,a} \mathcal{R}_K^L \mathcal{R}_L^H & \xleftarrow{c_{H,a} \mathbf{R}_{K,H}^L} & c_{L,a} \mathcal{R}_L^H \\
 \downarrow (\mathbf{C}_{L,a}^K)_{\mathcal{R}_L^H} & & \searrow \mathbf{C}_{H,a}^L \\
 \mathcal{R}_{a_K}^{a_L} c_{L,a} \mathcal{R}_L^H & \xrightarrow{\mathcal{R}_{a_L}^{a_R} \mathbf{C}_{H,a}^L} & \mathcal{R}_{a_K}^{a_L} \mathcal{R}_{a_L}^{a_H} c_{H,a} \\
 & & \nwarrow (\mathbf{R}_{a_K, a_H}^{a_L})_{c_{K,a}}
 \end{array}$$

(Diagram RRC)



$$\begin{array}{ccc}
c_{L,ab} \mathcal{R}_L^H & \xrightarrow{\mathbf{CR}_{L,H}^{ab}} & \mathcal{R}_{abL}^{abH} c_{H,ab} \\
\downarrow (\mathbf{C}_{a,b}^L) \mathcal{R}_L^H & & \searrow \mathcal{R}_{abL}^{abH} (\mathbf{C}_{a,b}^L) \\
& & \mathcal{R}_{abL}^{abH} c_{bL,a} c_{L,b} \\
& & \nearrow (\mathbf{CR}_{bL, bH}^a)_{c_{L,b}} \\
c_{bL,a} c_{L,b} \mathcal{R}_L^H & \xrightarrow{c_{bL,a} (\mathbf{CR}_{H,b}^L)} & c_{bL,a} \mathcal{R}_{bL}^{bH} c_{L,b}
\end{array}$$

(Diagram RCC)

$$\begin{array}{ccc}
c_{H,a} \mathcal{I}_L^H \mathcal{I}_K^L & \xleftarrow{c_{H,a} (\mathbf{R}_{K,H}^L)} & c_{H,a} \mathcal{I}_K^H \\
\downarrow (\mathbf{CI}_{H,a}^L) \mathcal{I}_K^L & & \searrow \mathbf{CI}_{H,a}^K \\
& & \mathcal{I}_{aK}^{aH} c_{K,a} \\
& & \nearrow (\mathbf{R}_{aK, aH}^{aL})_{c_{K,a}} \\
\mathcal{I}_{aL}^{aH} c_{L,a} \mathcal{I}_K^L & \xrightarrow{\mathcal{I}_{aL}^{aH} (\mathbf{CI}_{L,a}^K)} & \mathcal{I}_{aL}^{aH} \mathcal{I}_{aK}^{aL} c_{K,a}
\end{array}$$

(Diagram IIC)

$$\begin{array}{ccc}
c_{H,ab} \mathcal{I}_L^H & \xrightarrow{\mathbf{CI}_{H,ab}^L} & \mathcal{I}_{abL}^{abH} c_{L,ab} \\
\downarrow (\mathbf{C}_{a,b}^H) \mathcal{I}_L^H & & \searrow \mathcal{I}_{abL}^{abH} (\mathbf{C}_{a,b}^L) \\
& & \mathcal{I}_{abL}^{abH} c_{bL,a} c_{L,b} \\
& & \nearrow (\mathbf{CI}_{bH,a}^{bL})_{c_{L,b}} \\
c_{bH,a} c_{H,b} \mathcal{I}_L^H & \xrightarrow{c_{bH,a} (\mathbf{CI}_{H,b}^L)} & c_{bH,a} \mathcal{I}_{bL}^{bH} c_{L,b}
\end{array}$$

(Diagram ICC)

**3.3. Categorical Green functors.** A *categorical Green functor* over  $\mathcal{S}$ , is a Mackey-functor  $\mathcal{M}$  such that

- (CG0)  $\mathcal{M}(L)$  is a unitary monoidal category,
- (CG1)  $\mathcal{R}_L^K$  and  $c_{L,g}$  are always unitary monoidal functors,
- (CG2)  $\mathcal{I}_L^K$  is a  $\mathcal{M}(K)$ -left module functor,

- (CG3)  $\mathcal{I}_L^K$  is a  $\mathcal{M}(K)$ -right module functor,  
 (CG4) For any two subgroups  $L, K \leq H$  one has that

$$(3.6) \quad \mathbf{RI}_{L,K}^H : \mathcal{R}_L^H \circ \mathcal{I}_K^H \xrightarrow{\cong} \bigoplus_{x \in L \setminus H/K} \mathcal{I}_{xK \cap L}^L \circ \mathcal{R}_{xK \cap L}^{xK} \circ c_{K,x}.$$

as  $\mathcal{M}(G)$ -module functors.

- (CG5) The natural transformations  $\mathbf{R}_{K,H}^L, \mathbf{C}_{a,b}^H$  are natural transformation of monoidal functors.  
 (CG6)  $\mathbf{I}_{J,H}^K$  is a natural morphism  $\mathcal{M}(H)$ -module functors.  
 (CG7) and  $\mathbf{CI}_{H,a}^K, \mathbf{CR}_{H,a}^K$  are natural morphisms of  $\mathcal{M}(G)$ -module functors.

#### 4. CATEGORICAL MACKEY FUNCTORS FROM GROUP ACTIONS ON ABELIAN CATEGORIES

The goal of this section is to prove the main results mentioned in the introduction.

**Proposition 4.1.** *Let  $G$  be a finite group acting on the  $k$ -linear category  $\mathcal{C}$ . Let  $H \leq G$  be any subgroup of  $G$  and  $x \in G$ .*

- 1) *There is a  $k$ -linear functor  $c_{H,x} : \mathcal{C}^H \rightarrow \mathcal{C}^{xH}$  which is a  $k$ -linear equivalence of categories.*
- 2) *If  $M = (V, \mu) \in \mathcal{C}^H$  then  $c_{H,x}(M) := (T^x(V), {}^x\mu) \in \mathcal{C}^{xH}$  with the equivariant structure  $[{}^x\mu]_{T^x(V)}^{xhx^{-1}} : T^{xhx^{-1}}(T^x(V)) \rightarrow T^x(V)$  given by*

$$(4.2) \quad T^{xhx^{-1}}(T^x(V)) \xrightarrow{(T_2^{xhx^{-1}}, {}^x)_V} T^{xh}(V) \xrightarrow{(T_2^{x,h})_V^{-1}} T^x(T^h(V)) \xrightarrow{T^x(\mu_V^h)} T^x(V),$$

for any  $h \in H$ .

- 3) *If  $\mathcal{C}$  is a  $k$ -linear monoidal category and the action of  $G$  on  $\mathcal{C}$  is by monoidal autoequivalences then  $c_{H,x}$  is a  $k$ -linear monoidal functor.*

*Proof.* 1-2) In order to see that  $c_{H,x}(M)$  is an  ${}^xH$ -equivariant object it is enough to verify Equation (2.8) for any  $xhx^{-1}, xlx^{-1} \in {}^xH$ . This is equivalent to the diagram below (made of solid arrows) being commutative. Note that compatibility conditions (2.4)-(2.8) of the action of  $G$  imply the commutativity of the diagram after inserting the dashed arrows. Indeed, the bottom right trapeze (5) is commutative by applying  $T^x$  to the equivariantized condition (2.4) for  $V \in \mathcal{C}^H$ . The adjacent trapeze (6) is commutative by the naturality of  $T_2^{x,h}$  with respect to the morphism  $\mu_V^l$ . The rectangle (4) is commutative due to the associativity of the action, Equation (2.4). The parallelogram (2) is commutative due to the associativity of the action, Equation (2.4). Diagram (3) is commutative due to the naturality of the natural transformation  $T_2^{xhx^{-1},x}$  with respect to the morphism  $T^l(V) \xrightarrow{\mu_V^l} V$ . Diagram (1) is commutative due to the associativity of the action, Equation (2.4).

$$\begin{array}{ccccc}
T^{xhx^{-1}}(T^{xhx^{-1}}(T^x(V))) & \xrightarrow{T^{xhx^{-1}}(T_2^{xhx^{-1},x})_V} & T^{xhx^{-1}}(T^{xl}(V)) & \xrightarrow{T^{xhx^{-1}}(T_2^{x,l})_V} & T^{xhx^{-1}}(T^x(T^l(V))) \\
\downarrow (T_2^{xhx^{-1},xlx^{-1}})_V & & \downarrow (T_2^{xhx^{-1},xl})_V(T_2^{xhx^{-1},x})_{T^l(V)} & & \downarrow T^{xhx^{-1}}(T^x(\mu_V^l)) \\
T^{xhlx^{-1}}(T^x(V)) & & T^{xh}(T^l(V)) & & T^{xh}(V) \\
\downarrow (T_2^{xhlx^{-1},x})_V & & \downarrow (T_2^{x,h})_{T^l(V)} & & \downarrow (T_2^{x,h})_V \\
T^{xhl}(V) & \xrightarrow{(T_2^{x,h,l})_V^{-1}} & T^{xh}(T^l(V)) & \xrightarrow{T^{xh}(\mu_V^l)} & T^{xh}(V) \\
\downarrow (T_2^{x,h,l})_V^{-1} & & \downarrow (T_2^{x,h})_{T^l(V)} & & \downarrow (T_2^{x,h})_V \\
T^x(T^{hl}(V)) & \xrightarrow{T^x((T_2^{h,l})_V)} & T^x(T^h(T^l(V))) & & T^x(T^h(V)) \\
\downarrow T^x(\mu_V^{hl}) & & \downarrow T^x(T^h(\mu_V^l)) & & \downarrow T^x(\mu_V^h) \\
T^x(V) & \xleftarrow{T^x(\mu_V^h)} & T^x(T^h(V)) & & T^x(T^h(V))
\end{array}$$

(1) (2) (3) (4) (5) (6)

It is easy also to verify that if  $f : M \rightarrow N$  is a morphism in  $\mathcal{C}^H$  then  $T^x(f)$  is a morphism in  $\mathcal{C}^{xH}$ . Then it is clear that  $c_{H,x}$  is an equivalence of categories with the inverse given by  $c_{xH,x^{-1}} : \mathcal{C}^{xH} \rightarrow \mathcal{C}^H$ .

2) Let  $M, N$  be two objects of  $\mathcal{C}^H$ . If the action of  $G$  on  $\mathcal{C}$  is by monoidal autoequivalences then one can consider  $(T_2^x)^{M,N}$  as the monoidal structure of  $c_{H,x} : c_{H,x}(M \otimes N) \rightarrow c_{H,x}(M) \otimes c_{H,x}(N)$ . It is clear that  $(T_2^x)^{M,N}$  satisfies the pentagon axiom from the definition of a monoidal structure.

One has to check that the monoidal structure  $(T_2^x)^{M,N} : T^x(M \otimes N) \rightarrow T^x(M) \otimes T^x(N)$  of  $T^x$  is a morphism in  $\mathcal{C}^{xH}$ . Thus for any  $M, N \in \mathcal{C}^{xH}$  one has to check the commutativity of the following diagram:

$$\begin{array}{ccc}
T^{xhx^{-1}}(T^x(M \otimes N)) & \xrightarrow{T^{xhx^{-1}}((T_2^x)^{M,N})} & T^{xhx^{-1}}(T^x(M) \otimes T^x(N)) \\
\downarrow (T_2^{xhx^{-1},x})_{M \otimes N} & (1) & \downarrow (T_2^{xhx^{-1}})^{T^x(M), T^x(N)} \\
T^{xh}(M \otimes N) & \xrightarrow{(T_2^{xh})^{M,N}} & T^{xh}(M) \otimes T^{xh}(N) \\
\downarrow (T_2^{x,h})_{M \otimes N}^{-1} & (2) & \downarrow (T_2^{x,h})_M^{-1} \otimes (T_2^{x,h})_N^{-1} \\
T^x(T^h(M \otimes N)) & \xrightarrow{(T_2^x)^{T^h(M), T^h(N)}} & T^x(T^h(M)) \otimes T^x(T^h(N)) \\
\downarrow T^x((T_2^h)^{M,N}) & (3) & \downarrow T^x(\mu_h^M) \otimes T^x(\mu_h^N) \\
T^x(T^h(M) \otimes T^h(N)) & \xrightarrow{(T_2^x)^{M,N}} & T^x(M) \otimes T^x(N)
\end{array}$$

The upper pentagon (1) is commutative since  $T_2^{xax^{-1},x}$  is a natural transformation of monoidal functors, Equation (2.14). The middle pentagon (2) is commutative since  $T_2^{x,a}$  is a natural transformation of monoidal functors, same Equation (2.14). The bottom rectangle commutes from the compatibility condition of the monoidal functor  $T^x$  with the tensor product of morphisms, Equation (2.1).  $\square$

**Remark 4.3.** Note that if  $x \in H$  then  $c_{H,x} \simeq \text{id}_{\mathcal{C}^H}$  via the natural transformation  $N_V : c_{H,x}(V) \xrightarrow{\mu_V^x} V$  for any  $(V, \mu) \in \mathcal{C}^H$ . Moreover, if  $\mathcal{C}$  is a  $k$ -linear monoidal category and the action of  $G$  on  $\mathcal{C}$  is by monoidal autoequivalences then  $c_{H,x} \simeq \text{id}_{\mathcal{C}^H}$  as monoidal functors.

**Proposition 4.4.** Let  $G$  be a finite group acting  $k$ -linearly on a  $k$ -linear category  $\mathcal{C}$ . With the above notations there is a natural transformation which is an isomorphism of  $k$ -linear functors

$$\mathbf{C}_{a,b}^H : c_{H,ab} \rightarrow c_{bH,a} c_{H,b}$$

If  $\mathcal{C}$  is a  $k$ -linear monoidal category and the action of  $G$  on  $\mathcal{C}$  is by monoidal autoequivalences then  $\mathbf{C}_{a,b}^H$  is an isomorphism of monoidal functors.

*Proof.* One can easily check that  $(\mathbf{C}_{a,b}^H)_M := (T_2^{a,b})_M^{-1} : T^{ab}(M) \rightarrow T^a(T^b(M))$  is a morphism in  $\mathcal{C}^{abH}$ . Indeed for any  $h \in H$  one has to check that the following diagram is commutative:

$$\begin{array}{ccc}
T^{abh(ab)^{-1}}(T^{ab}(M)) & \xrightarrow{T^{abh(ab)^{-1}}((T_2^{a,b})_M^{-1})} & T^{abh(ab)^{-1}}(T^a(T^b(M))) \\
\downarrow (T_2^{abh(ab)^{-1},ab})_M & & \downarrow (T_2^{abh(ab)^{-1},a})_{T^b(M)} \\
T^{abh}(M) & \xrightarrow{\quad\quad\quad} & T^{abh^{-1}}(T^b(M)) \\
\downarrow (T_2^{ab,h})_M^{-1} & \searrow & \downarrow (T_2^{abh^{-1},b})_M \\
T^{ab}(T^h(M)) & & T^a(T^{bh^{-1}}(T^b(M))) \\
\downarrow T^{ab}(\mu_M^h) & \searrow & \downarrow T^a((T_2^{bh^{-1},b})_M) \\
T^{ab}(M) & & T^a(T^{bh}(M)) \\
& \searrow (T_2^{a,b})_M^{-1} & \downarrow T^a((T_2^{b,h})_M^{-1}) \\
& & T^a(T^b(T^h(M))) \\
& & \downarrow T^a(T^b(\mu_M^h)) \\
& & T^a(T^b(M))
\end{array}$$

Note that the last parallelogram is commutative by the naturality of  $T_2^{a,b}$ . All the other parallelograms are commutative by Equation (2.8). This defines a natural transformation  $\mathbf{C}_{a,b}^H : c_{H,ab} \rightarrow c_{bH,a}c_{H,b}$  which clearly it is an isomorphism.

If the action is by monoidal equivalences then it is clear that  $\mathbf{C}_{a,b}^H$  is a natural morphisms of monoidal functors. Indeed  $\mathbf{C}_{a,b}^H$  satisfies Equation (2.3) since  $T_2^{a,b} : T^a T^b \rightarrow T^{ab}$  is also a morphism of monoidal functors.  $\square$

**4.1. Module category structures.** Suppose that the group  $G$  acts by monoidal equivalences on the  $k$ -linear monoidal category  $\mathcal{C}$ . For any  $L \leq H \leq G$  note that  $\mathcal{C}^L$  is a  $\mathcal{C}^H$ -left (or right) module category via the restriction functor  $\text{Res}_H^L$ . Thus one can define  $M \boxtimes (V, \mu) := (\text{Res}_H^L(M) \otimes V, \nu)$  where for any  $h \in H$  one has that  $\nu_{M \otimes V}^h : T^h(M \otimes V) \rightarrow M \otimes V$  is defined via:

$$\nu_{M \otimes V}^h : T^h(M \otimes V) \xrightarrow{(T_2^h)^{M,V}} T^h(M) \otimes T^h(V) \xrightarrow{\mu_V^h \otimes \mu_M^h} M \otimes V.$$

Note also that  $M \boxtimes V = \text{Res}_H^G(M) \otimes V$  and the  $\mathcal{C}^L$ -module category associativity constraint of  $\mathcal{C}^H$  coincides to the associativity constraint of  $\mathcal{C}$  as a monoidal category.

**4.1.1. On the module functor structures of the restriction and induction functors.** If  $L \leq H$  are subgroups of  $G$  then the restriction functor  $\text{Res}_L^H : \mathcal{C}^H \rightarrow \mathcal{C}^L$  is a  $\mathcal{C}^G$ -module functor. Indeed note that  $\text{Res}_L^H(V \boxtimes M) = \text{Res}_L^H(\text{Res}_H^G(M) \otimes V) = \text{Res}_L^H(\text{Res}_H^G(M)) \otimes \text{Res}_L^H(V) = \text{Res}_L^G(M) \otimes \text{Res}_L^H(V) = M \boxtimes \text{Res}_L^H(V)$ . Thus the module functor structure of the functor  $\text{Res}_L^H$  can be considered as the identity map. Moreover, since for any  $K \leq L \leq H$  one has  $\text{Res}_K^L \text{Res}_L^H = \text{Res}_K^H$  it follows that one can also consider  $\mathbf{R}_{K,H}^L : \text{Res}_K^L \text{Res}_L^H \rightarrow \text{Res}_K^H$  as the identity natural transformation.

Let  $(V, \mu) \in \mathcal{C}^H$  and  $L \leq H$ . By abuse of notations sometimes we write  $\text{Ind}_H^G(V)$  instead of  $\text{Ind}_H^G(V, \mu)$ . We also write shortly  $\text{Res}_L^H(V)$  instead of  $\text{Res}_L^H(V, \mu)$ .

**Lemma 4.5.** *Let  $G$  be a finite group acting by monoidal autoequivalences on  $\mathcal{C}$  and  $L \leq H \leq G$  be a tower of subgroups. With the above notations the induction functor  $\text{Ind}_L^H : \mathcal{C}^L \rightarrow \mathcal{C}^H$  is a  $\mathcal{C}^H$ -module functor.*

*Proof.* Let  $M \in \mathcal{C}^H$  and  $V \in \mathcal{C}^L$ . We will define a canonical isomorphism

$$(4.6) \quad (\text{Ind}_L^H)_2^{M,V} : \text{Ind}_L^H(M \boxtimes V) \rightarrow M \boxtimes \text{Ind}_L^H(V)$$

in  $\mathcal{C}^H$  and we will show that it satisfies the module functor axioms. Fix a set  $\mathcal{R}$  of representative elements for the left cosets of  $H \leq L$ . Let  $\text{Ind}_H^L$  be the associated induction functor  $\text{Ind}_H^L : \mathcal{C}^L \rightarrow \mathcal{C}^H$  to the set  $\mathcal{R}$ . Thus

$$\text{Ind}_H^L(M \boxtimes V) = \text{Ind}_H^L(\text{Res}_H^L(M) \otimes V) = \oplus_{a \in \mathcal{R}} T^a(M \otimes V)$$

and

$$M \boxtimes \text{Ind}_L^H(V) = \text{Res}_H^G(M) \otimes (\oplus_{a \in H/L} T^a(V)) = \oplus_{a \in H/L} M \otimes T^a(V)$$

On the components, for any  $a \in H$  the module functor structure will be defined as

$$(\text{Ind}_L^H)_2^{M,V,a} : T^a(M \otimes V) \xrightarrow{(T_2^a)^{M,V}} T^a(M) \otimes T^a(V) \xrightarrow{\mu_M^a \otimes 1} M \otimes T^a(V).$$

One needs to verify that the above module functor structure is a morphism in  $\mathcal{C}^H$  which is equivalent to the following commutative diagram:

$$\begin{array}{ccc} T^h(\text{Ind}_L^H(M \boxtimes V)) & \xrightarrow{T^h((\text{Ind}_L^H)_2^{M,V})} & T^h(M \boxtimes \text{Ind}_L^H(V)) \\ \downarrow \mu_{\text{Ind}_L^H(M \boxtimes V)}^h & & \downarrow \mu_{M \boxtimes \text{Ind}_L^H(V)}^h \\ \text{Ind}_L^H(M \boxtimes V) & \xrightarrow{(\text{Ind}_L^H)_2^{M,V}} & M \boxtimes \text{Ind}_L^H(V). \end{array}$$

On the components this can be written as the commutativity of the following:

$$\begin{array}{ccccc}
T^h(T^a(M \otimes V)) & \xrightarrow{T^h[(T_2^a)^{M,V}]} & T^h(T^a(M) \otimes T^a(V)) & \xrightarrow{T^h(\mu_M^a 1)} & T^h(M \otimes T^a(V)) \\
\downarrow (T_2^{h,x})_{T^r(M)} & & \downarrow (T_2^h)^{T^a(M), T^a(V)} & & \downarrow (T_2^h)^{M, T^a(V)} \\
& (1) \quad T^h(T^a(M)) \otimes T^h(T^a(V)) & & & \\
& \downarrow (T_2^{h,a})_M (T_2^{h,a})_V & \searrow T^h(\mu_M^a) \Gamma & & \\
T^{ha}(M \otimes V) & \xrightarrow{(T_2^{ha})^{M,V}} & T^{ha}(M) \otimes T^{ha}(V) & & T^h(M) \otimes T^h(T^a(V)) \\
\downarrow (T_2^{x,a})_{T^r(M)}^{-1} & & \downarrow (T_2^{b,l})_M^{-1} \otimes (T_2^{b,l})_V^{-1} & & \downarrow \mu_M^h 1 \\
& (3) & & (4) & \\
& (T_2^{b,l})_M^{-1} \otimes (T_2^{b,l})_V^{-1} & \searrow \mu_M^{ha} 1 & & \\
T^b(T^l(M \otimes V)) & & & & M \otimes T^h(T^a(V)) \\
\downarrow T^b((T_2^l)^{M,V}) & & & & \downarrow 1(T_2^{h,a})_V \\
& (T_2^b)^{T^l(M), T^l(V)} & \xrightarrow{(T_2^b)^{T^l(M), T^l(V)}} & T^b(T^l(M)) \otimes T^b(T^l(V)) & (6) \quad M \otimes T^{ha}(V) \\
\downarrow T^b(\mu_M^l \otimes \mu_V^l) & & & & \downarrow M((T_2^{b,l})_V^{-1}) \\
& T^b(M \otimes V) & \xrightarrow{T^b(M) \otimes T^b(T^l(V))} & M \otimes T^b(T^l(V)) & \\
\downarrow (T_2^b)^{M,V} & & \downarrow \mu_M^b \otimes 1 & & \downarrow 1T^b(\mu_V^l) \\
& T^b(M) \otimes T^b(V) & \xrightarrow{\mu_M^b \otimes 1} & M \otimes T^b(V) &
\end{array}$$

Using Equation (2.8) it is straightforward to verify that the above map  $(\text{Ind}_L^H)_2^{M,V}$  is a module functor structure.  $\square$

**Remark 4.7.** Note that the induction functor defined above depends on the set  $\mathcal{R}$  of chosen representative elements for the left cosets  $L \leq H$ . Moreover, as explained in Section 2.3 changing the representative elements one obtains a  $k$ -linear isomorphic functor. Since  $\text{Res}_L^H$  is a  $\mathcal{C}^H$ -module functor and  $\text{Ind}_H^L$  is its left adjoint it follows that  $\text{Ind}_L^H$  as a  $\mathcal{C}^H$ -module functor does not depend on the chosen set  $\mathcal{R}$ .

**Lemma 4.8.** Let  $G$  be a finite group acting on the abelian category  $\mathcal{C}$ .

1) Suppose that  $L \leq H$  are two subgroups of  $G$ . Then the following identities hold for any  $x \in G$ :

$$(4.9) \quad c_{L,x} \circ \text{Res}_L^H = \text{Res}_{xL}^H \circ c_{H,x}$$

as functors from  $\mathcal{C}^L$  to  $\mathcal{C}^{xH}$ .

2) There is a natural transformation

$$(4.10) \quad \mathbf{CI}_{L,H}^x : c_{H,x} \circ \text{Ind}_L^H \xrightarrow{\simeq} \text{Ind}_{xL}^H \circ c_{L,x}$$

which is a natural isomorphism of functors from  $\mathcal{C}^L$  to  $\mathcal{C}^{xH}$

3) If  $\mathcal{C}$  is a  $k$ -linear monoidal category and the action of  $G$  on  $\mathcal{C}$  is by monoidal autoequivalences then  $\mathbf{CI}_{L,H}^x$  is an isomorphism of  $\mathcal{C}^G$ -module functors.

*Proof.* 1) The first identity is straightforward.

2) Let  $\mathcal{R}$  be a set of representative elements for the left cosets  $\{Lx \mid x \in H\}$  of the extension  $H/L$ . Suppose that

$$(4.11) \quad \text{Ind}_L^H(M) = (\oplus_{r \in \mathcal{R}} T^r(M), \nu_{\text{Ind}_L^H(M)})$$

for any  $(M, \{\mu_M^b\}_{b \in L}) \in \mathcal{C}^L$ . Then using formula (2.12) it follows that  $\nu_{\text{Ind}_L^H(M)}^a$  is defined on the components as follows:

$$(4.12) \quad \nu^{a,r} : T^a(T^r(M)) \xrightarrow{T_2^{a,r}(M)} T^{ar}(M) \xrightarrow{(T_2^{r',b})_M^{-1}} T^{r'}(T^b(M)) \xrightarrow{T^{r'}(\mu_M^b)} T^{r'}(M)$$

where  $r' \in \mathcal{R}$  and  $b \in L$  are determined by  $ar = r'b$ . On the other hand, using Proposition 4.1 for  $c_{H,x}(\text{Ind}_L^H(M))$ , as an object of  $\mathcal{C}^{xH}$  one has that

$$(4.13) \quad c_{H,x}(\text{Ind}_L^H(M)) = (\oplus_{r \in \mathcal{R}} T^x(T^r(M)), {}^x\nu_{T^x(\text{Ind}_L^H(M))})$$

with the equivariant structure  $[{}^x\nu]$  given on the components by:

$$\begin{aligned} [{}^x\nu]^{xax^{-1},r} : T^{xax^{-1}}(T^x(T^r(M))) &\xrightarrow{(T_2^{h,x})_{T^r(M)}} T^{xa}(T^r(M)) \xrightarrow{(T_2^{x,a})^{-1}_{T^r(M)}} \\ &\xrightarrow{(T_2^{x,a})^{-1}_{T^r(M)}} T^x(T^a(T^r(M))) \xrightarrow{T^x(T_2^{a,r})_M} T^x(T^{ar}(M)) \xrightarrow{T^x(T_2^{r',b})_M^{-1}} \\ &\xrightarrow{T^x(T_2^{r',b})_M^{-1}} T^x(T^{r'}(T^b(M))) \xrightarrow{T^x(T^{r'}(\mu_M^b))} T^x(T^{r'}(M)) \end{aligned}$$

On the other hand, since  $x\mathcal{R}x^{-1}$  is a set of representative for the left cosets of  ${}^xH/{}^xL$ , by Remark 4.7 one may suppose that

$$(4.14) \quad \text{Ind}_{{}^xL} {}^xH c_{L,x}(M) = (\oplus_{r \in \mathcal{R}} T^{xrx^{-1}}(T^x(M)), \eta_{\text{Ind}_{{}^xL} {}^xH T^x(M)})$$

Using again formula (2.12) it follows that the equivariant structure of  $\text{Ind}_{{}^xL} {}^xH T^x(M)$  on the components is given as follows:

$$\begin{aligned} \eta_M^{xax^{-1}, xrx^{-1}} : T^{xax^{-1}}(T^{xrx^{-1}}(T^x(M))) &\xrightarrow{(T_2^{xax^{-1}, xrx^{-1}})_{T^x(M)}} \\ T^{xarx^{-1}}(T^x(M)) &\xrightarrow{(T_2^{xrx^{-1}, xbx^{-1}})^{-1}_{T^x(M)}} T^{xrx^{-1}}(T^{xbx^{-1}}(T^x(M))) \\ &\xrightarrow{T^{xrx^{-1}}(({}^x\mu)^{xbx^{-1}})} T^{xrx^{-1}}(T^x(M)) \end{aligned}$$

Define the natural transformation  $\mathbf{CI}_{L,H}^x := \oplus_{r \in \mathcal{R}} (T_2^{xrx^{-1}, x})^{-1}(T_2^{x,r}) : c_{H,x} \text{Ind}_L^H \rightarrow \text{Ind}_{{}^xL} {}^xH c_{L,x}$ . It will be shown that  $(\mathbf{CI}_{L,H}^x)_M : T^x(\text{Ind}_L^H(M)) \rightarrow \text{Ind}_{{}^xL} {}^xH (T^x(M))$  is an isomorphism in  $\mathcal{C}^{xH}$  for any  $M \in \mathcal{C}^L$ . Indeed, one has to verify that the above morphism  $\mathbf{CI}_{L,H}^x$  is compatible with the two equivariant structures defined above. This means that the following diagram: is commutative.

$$\begin{array}{ccc} T^{xax^{-1}}(T^x \text{Ind}_L^H(M)) & \xrightarrow{T^{xax^{-1}}((\mathbf{CI}_{L,H}^x)_M)} & T^{xax^{-1}}(\text{Ind}_{{}^xL} {}^xH T^x(M)) \\ \downarrow ({}^x\nu^{xax^{-1}})_M & & \downarrow \eta_M^{xax^{-1}} \\ T^x(\text{Ind}_L^H(M)) & \xrightarrow{(\mathbf{CI}_{L,H}^x)_M} & \text{Ind}_{{}^xL} {}^xH T^x(M). \end{array}$$

On the components the above diagram becomes the following:



$$\begin{array}{ccccc}
T^{xax^{-1}}(T^x(T^r(M))) & \xrightarrow{T^{xax^{-1}}(T_2^{x,r})_M} & T^{xax^{-1}}(T^{xr}(M)) & \xrightarrow{(T_2^{x,a})_{T^r(M)}^{-1}} & T^{xax^{-1}}(T^{xr x^{-1}}(T^x(M))) \\
\downarrow (T_2^{xax^{-1},x})_{T^r(M)} & & \downarrow (T_2^{xax^{-1},xr})_{T^r(M)} & & \downarrow (T_2^{xax^{-1},xr x^{-1}})_{T^x(M)} \\
T^{xa}(T^r(M)) & \xrightarrow{(T_2^{x,a,r})_M} & T^{xar}(M) & \xrightarrow{(T_2^{x,r'})_{T^b(M)}^{-1}} & T^{xar x^{-1}}(T^x(M)) \\
\downarrow (T_2^{x,a})_{T^r(M)}^{-1} & & \downarrow (T_2^{x,r'})_{T^b(M)} & & \downarrow (T_2^{x,r' x^{-1},xb x^{-1}})_{T^x(M)}^{-1} \\
T^x(T^a(T^r(M))) & \xrightarrow{(T_2^{x,r'b})_{T^b(M)}^{-1}} & T^{xr'b}(M) & \xrightarrow{(T_2^{x,r' x^{-1},xb})_{T^b(M)}^{-1}} & T^{xr' x^{-1}}(T^{xb x^{-1}}(T^x(M))) \\
\downarrow T^x((T_2^{a,r})_M^{-1}) & & \downarrow T^{xr' x^{-1}}((T_2^{xb x^{-1},x})_M) & & \downarrow T^{xr' x^{-1}}(T^{xb}(M)) \\
T^x(T^{ar}(M)) & \xrightarrow{(T_2^{x,r'})_{T^b(M)}} & T^{xr'b}(M) & \xrightarrow{(T_2^{x,r' x^{-1},x})_{T^b(M)}^{-1}} & T^{xr' x^{-1}}(T^x(T^b(M))) \\
\downarrow T^x((T_2^{r',b})_M^{-1}) & & \downarrow T^{xr' x^{-1}}((T_2^{x,b})_M^{-1}) & & \downarrow T^{xr' x^{-1}}(T^x(\mu_M^b)) \\
T^x(T^{r'}(T^b(M))) & \xrightarrow{(T_2^{x,r'})_{T^b(M)}} & T^{xr'}(T^b(M)) & \xrightarrow{(T_2^{x,r' x^{-1},x})_{T^b(M)}^{-1}} & T^{xr' x^{-1}}(T^x(T^b(M))) \\
\downarrow T^x(T^{r'}(\mu_M^b)) & & \downarrow T^{xr'}(\mu_M^b) & & \downarrow T^{xr' x^{-1}}(T^x(\mu_M^b)) \\
T^x(T^{r'}(M)) & \xrightarrow{(T_2^{x,r'})_{T^b(M)}} & T^{xr'}(M) & \xrightarrow{(T_2^{x,r' x^{-1},x})_M^{-1}} & T^{xr' x^{-1}}(T^x(M))
\end{array}$$

(1) (2) (3) (4) (5) (6)

Note that diagrams (1) – (6) are commutative by the associativity of the action, Equation (2.4). The bottom two rectangles are commutative since  $T_2^{x,r'}$  and  $T_2^{xr' x^{-1},x}$  are natural transformations.

3) It is also straightforward to verify that this defines a natural transformation of  $\mathcal{C}^G$ -module functors in the case of an action by monoidal autoequivalences. Indeed let  $M \in \mathcal{C}^G$  and  $V \in \mathcal{C}^H$ . One has to check that the following diagram is commutative:

$$\begin{array}{ccc}
c_{L,x} \text{Ind}_H^L(M \otimes V) & \xrightarrow{(\mathbf{CI}_{L,H}^x)_{M \otimes V}} & \text{Ind}_H^x c_{H,x}(M \otimes V) \\
\downarrow (c_{L,x} \text{Ind}_H^L)^{M,V}_2 & & \downarrow (\text{Ind}_H^x c_{H,x})^{M,V}_2 \\
M \otimes c_{L,x} \text{Ind}_H^L(V) & \xrightarrow{1 \otimes (\mathbf{CI}_{L,H}^x)_V} & M \otimes \text{Ind}_H^x c_{H,x}(V)
\end{array}$$

On the components the above diagram is equivalent to the commutativity of the following diagram; note that for shortness we replaced the symbol “ $\otimes$ ” by “.”.

$$\begin{array}{ccccc}
T^x(T^r(M.V)) & \xrightarrow{(T_2^{x,r})^{M,V}} & T^{xr}(M.V) & \xrightarrow{(T_2^{xrx^{-1},x})_{M,V}^{-1}} & T^{xrx^{-1}}(T^x(M.V)) \\
\downarrow T^x((T_2^r)^{M,V}) & & \downarrow (T_2^{xr})^{M,V} & & \downarrow T^{xrx^{-1}}((T_2^x)^{M,V}) \\
T^x(T^r(M).T^r(V)) & \xrightarrow{(T_2^{x,r})^{-1} \cdot (T_2^{x,r})^{-1}} & T^{xr}(M).T^{xr}(V) & \xrightarrow{(T_2^{xrx^{-1},x})_M^{-1} \cdot (T_2^{xrx^{-1},x})_V^{-1}} & T^{xrx^{-1}}(T^x(M).T^x(V)) \\
\downarrow T^x(\mu_M^r \cdot 1) & & \downarrow T^x(\mu_M^r \cdot 1) & & \downarrow T^{xrx^{-1}}(\mu_M^x \cdot 1) \\
T^x(M.T^r(V)) & \xrightarrow{(T_2^x)^{M,T^r(V)}} & T^x(M).T^x(T^r(V)) & \xrightarrow{(T_2^{xrx^{-1},x})_M^{-1} \cdot (T_2^{xrx^{-1},x})_V^{-1}} & T^{xrx^{-1}}(M.T^x(V)) \\
\downarrow \mu_M^x \cdot 1 & & \downarrow \mu_M^x \cdot 1 & & \downarrow \mu_M^{xrx^{-1}} \cdot 1 \\
M.T^x(T^r(V)) & \xrightarrow{1 \cdot (T_2^{x,r})_V} & M.T^{xr}(V) & \xrightarrow{1 \cdot (T_2^{xrx^{-1},x})_V^{-1}} & M.T^{xrx^{-1}}(T^x(V))
\end{array}$$

(D1)      (D2)      (D3)      (D4)      (D5)      (D6)

The pentagon (D1) from the upper left corner is commutative by the fact that  $T_2^{x,r}$  is a natural isomorphism of monoidal functors. On the other hand the pentagon (D2) from the upper right corner is commutative by the fact that  $T_2^{xrx^{-1},x}$  is a natural isomorphism of monoidal functors. Note that the commutativity of the diagram (D3) follows by the naturality of the transformation  $T_2^x$ , i.e Equation (2.14). A similar argument applies for diagram (D4). Commutativity of the diagrams (D5) and (D6) follows by Equation (2.8).  $\square$

**Remark 4.15.** Note that by the first statement of the previous lemma the monoidal functor  $c_{H,x} : \mathcal{C}^H \rightarrow \mathcal{C}^{xH}$  also becomes a  $\mathcal{C}^G$ -module functor. Indeed, for any  $M \in \mathcal{C}^G$  and any  $V \in \mathcal{C}^H$  one has that

$$c_{H,x}(M \boxtimes V) = c_{H,x}(\text{Res}_H^G(M) \otimes V) \simeq c_{H,x}(\text{Res}_H^G(M)) \otimes c_{H,x}(V) = \text{Res}_H^G c_{G,x}(M) \otimes c_{H,x}(V)$$

On the other hand since  $c_{G,x} \simeq \text{id}_{\mathcal{C}^G}$  it follows that  $c_{H,x}(M \boxtimes V) \simeq M \boxtimes c_{H,x}(V)$ , which shows that  $c_{H,x}$  is a  $\mathcal{C}^G$ -module functor. Its module structure it is given by

$$(4.16) \quad T^x(M \otimes V) \xrightarrow{T_2^x} T^x(M) \otimes T^x(V) \xrightarrow{\mu_M^x \otimes 1} M \otimes T^x(V).$$

**4.2. Proof of Theorem 1.2.** We are now ready to give a proof for Theorem 1.2.

*Proof.* Fix a set  $\mathcal{D}$  of representative elements for the double cosets  $K \backslash H / L$ . For any  $x \in \mathcal{D}$  consider an arbitrary set  $\mathcal{R}_x$  of representative elements for the left cosets of  $yL$  of  $KxL/L$ . Since  $H = \sqcup_{x \in \mathcal{D}} KxL$  it follows that  $\mathcal{R} := \sqcup_{x \in \mathcal{D}} \mathcal{R}_x$  is a complete set of representative elements for the left cosets  $H/L$ . Suppose that  $(V, \mu_V) \in \mathcal{C}^L$ . Then the induction functor  $\text{Ind}_H^L$  associated to  $\mathcal{R}$  can be written as

$$\text{Ind}_L^H(V) \simeq \left( \bigoplus_{r \in \mathcal{R}} T^r(V), \mu_{\text{Ind}_H^G(V)} \right) = \left( \bigoplus_{x \in \mathcal{D}} \bigoplus_{a \in \mathcal{R}_x} T^a(V), \mu_{\text{Ind}_H^G(V)} \right)$$

where by Equation (2.12) the equivariant structure is given on components by

$$\mu_{\text{Ind}_L^H(M)}^{g,a} : T^g(T^a(V)) \xrightarrow{(T_2^{g,a})_V} T^{ga}(V) = T^{yh}(V) \xrightarrow{(T_2^{y,l})_V^{-1}} T^y T^l(V) \xrightarrow{T^y(\mu_V^l)} T^y(V)$$

if  $ga = yl$  with  $y \in \mathcal{R}$  and  $l \in L$ .

For any  $x \in \mathcal{D}$  let

$$(4.17) \quad F_x(V) := \bigoplus_{a \in \mathcal{R}_x} T^a(V).$$

It can be easily verified that the above induced equivariant structure  $\mu_{\text{Ind}_H^G(V)}$  of  $\text{Ind}_H^G(V)$  sends the component  $F_x(V)$  to itself. Indeed, note that for any  $a \in \mathcal{R}_x$  if  $m \in K$  and  $ma = bl$  with  $l \in L$  then  $b = mal^{-1} \in KaH = KxH$ . It follows that  $(F_x(V), \nu|_K) \in \mathcal{C}^K$  and then one can define a functor  $F_x : \mathcal{C}^L \rightarrow \mathcal{C}^K$ . Moreover it is easy to see that  $\text{Res}_K^H \text{Ind}_L^H \simeq \bigoplus_{x \in \mathcal{D}} F_x$  as  $k$ -linear functors.

Define also the functor  $G_x : \mathcal{C}^{xL} \rightarrow \mathcal{C}^K$  given by  $G_x = \text{Ind}_{K \cap xL}^K \circ \text{Res}_{xL \cap K}^{xL}$ . Then in order to finish the proof it is enough to show that for all  $x \in \mathcal{D}$  one has

$$(4.18) \quad F_x \simeq G_x \circ c_{L,x}.$$

Note that there is a bijection between the following sets of left cosets  $K/K \cap xL \xrightarrow{\phi} KxL/L$  given by  $a(K \cap xL) \mapsto axL$  whose inverse is given by  $aL \mapsto ax^{-1}(K \cap xL)$ . This enables us to write  $G_x(P) \simeq \bigoplus_{a \in \mathcal{R}_x} T^{ax^{-1}}(P)$ , for any  $P \in \mathcal{C}^{xL}$ . Under this isomorphism the  $K$ -equivariant structure of  $G_x(P)$  becomes

$$\begin{aligned} \mu_{G_x(P)}^m : T^m(T^{ax^{-1}}(P)) &\xrightarrow{(T_2^{m,ax^{-1}})_P} T^{max^{-1}}(P) = T^{bx^{-1}m'}(P) \rightarrow \\ &\xrightarrow{(T_2^{bx^{-1},m'})_P^{-1}} T^{bx^{-1}}(T^{m'}(P)) \xrightarrow{T^{bx^{-1}}(\mu_P^{m'})} T^{bx^{-1}}(P) \end{aligned}$$

where  $b \in \mathcal{R}_x$  is chosen such that  $max^{-1} = bx^{-1}m'$  with  $m' \in xL \cap K$ . Thus for any  $V \in \mathcal{C}^L$  the  $K$ -equivariant structure of  $G_x c_{L,x}(V)$  is given by

$$\begin{aligned} \mu_{G_x c_{L,x}(V)}^m : T^m(T^{ax^{-1}}(T^x(V))) &\xrightarrow{(T_2^{m,ax^{-1}})_{T^x(V)}} T^{max^{-1}}(T^x(V)) = T^{bx^{-1}m'}(T^x(V)) \rightarrow \\ &\xrightarrow{(T_2^{bx^{-1},m'})_{T^x(V)}^{-1}} T^{bx^{-1}}(T^{m'}(T^x(V))) \xrightarrow{T^{bx^{-1}}(T_2^{m',x})_V} T^{bx^{-1}}(T^{m'x}(V)) \rightarrow \\ &\xrightarrow{T^{bx^{-1}}((T_2^{x,x^{-1}m'x})_V^{-1})} T^{bx^{-1}}(T^x(T^{x^{-1}m'x}(V))) \xrightarrow{T^{bx^{-1}}(T^x(\mu_V^{x^{-1}m'x}))} T^{bx^{-1}}(T^x(V)). \end{aligned}$$

Define the natural transformation  $N_x : F_x \rightarrow G_x \circ c_{L,x}$  by

$$(N_x)_V : F_x(V) \xrightarrow{\bigoplus_{a \in \mathcal{R}_x} (T_2^{ax^{-1},x})_V^{-1}} G_x(c_{L,x}(V))$$

In order to show that  $N_x$  is well defined it is enough to check that it is compatible with the two equivariant structures of the objects  $F_x(V)$  and  $G_x c_{H,x}(V)$ . Thus one has to verify that the following diagram is commutative:

$$\begin{array}{ccc}
 T^m(F_x(V)) & \xrightarrow{\mu_{F_x(V)}^m} & F_x(V) \\
 \downarrow T^m((N_x)_V) & & \downarrow (N_x)_V \\
 T^m(G_x c_{L,x}(V)) & \xrightarrow{\mu_{G_x c_{L,x}(V)}^m} & G_x c_{L,x}(V)
 \end{array}$$

On the components the above diagram is equivalent to the following. Suppose that  $m \in K$  and  $a, b \in \mathcal{R}_x$  with  $max^{-1} = bx^{-1}m'$  for some  $m' \in K \cap {}^x L$ . Then  $ma = bx^{-1}m'x = bl$  with  $l = (x^{-1}m'x) \in L$ .

$$\begin{array}{ccccc}
 T^m(T^{ax^{-1}}(T^x(V))) & \xrightarrow{(T_2^{m, ax^{-1}})_{T^x(V)_1}} & T^{max^{-1}}(T^x(V)) & = & T^{bx^{-1}m'}(T^x(V)) \xrightarrow{(T_2^{bx^{-1}, m'})_{T^x(V)}} T^{bx^{-1}}(T^{m'}(T^x(V))) \\
 \downarrow T^m[(T_2^{ax^{-1}, x})_V] & & \swarrow (T_2^{max^{-1}, x})_V = (T_2^{bx^{-1}m', x})_V & & \downarrow T^{bx^{-1}}(T_2^{m', x})_V \\
 T^m(T^a(V)) & & & & \\
 \downarrow (T_2)^{m, a}_V & & & & \\
 T^{ma}(V) = T^{bl}(V) & \xrightarrow{(T_2^{bx^{-1}, xl})_V} & T^{bx^{-1}}(T^{m'x}(V)) = T^b(T^{xl}(V)) & & \\
 \downarrow (T_2^{b, l})_V^{-1} & & \downarrow T^{bx^{-1}}(T_2^{x, l})_V & & \\
 T^b(T^l(V)) & \xrightarrow{(T_2^{bx^{-1}, x})_{T^l(V)}^{-1}} & T^{bx^{-1}}(T^x(T^l(V))) & & \\
 \downarrow T^b(\mu_V^l) & & \downarrow T^b(T^x(\mu_V^l)) & & \\
 T^b(V) & \xleftarrow{(T_2^{bx^{-1}, x})_V^{-1}} & T^{bx^{-1}}(T^x(V)) & & 
 \end{array}$$

The bottom rectangle is commutative by the naturality of  $T_2^{bx^{-1}, x}$  with respect to the morphisms, Equation (2.6). The above rectangle is commutative due to Equation (2.4), the associativity of the action. The upper left diagram is commutative by the same reason. The upper right trapeze is commutative by associativity of the action, Equation (2.4). Clearly  $(N_x)_V$  is an isomorphism in  $\mathcal{C}$  and therefore it is an isomorphism in  $\mathcal{C}^K$ .

2) Suppose now that  $\mathcal{C}$  is a  $k$ -linear monoidal category and the action of  $G$  is by monoidal autoequivalences. It remains to show that the isomorphism from the statement is an isomorphism of  $\mathcal{C}^G$ -module functors. For this, it is enough to show that the above isomorphism  $N_x$  of Equation (4.18) is an isomorphism of  $\mathcal{C}^G$ -module functors. This is equivalent to the commutativity of the following diagram:

$$\begin{array}{ccc}
 G_x c_{H,x}(M \boxtimes V) & \xrightarrow{(G_x c_{H,x})_2^{M,V}} & M \boxtimes G_x c_{H,x}(V) \\
 \downarrow (N_x)_{M \boxtimes V} & & \downarrow 1_M \boxtimes (N_x)_V \\
 F_x(M \boxtimes V) & \xrightarrow{(F_x)_2^{M,V}} & M \boxtimes F_x(V).
 \end{array}$$

Note that the  $\mathcal{C}^G$ -module structure of  $F_x$  is the one induced from  $\text{Ind}_L^H$ . Thus one has that  $(F_x)_2^{M,V} : F_x(M \otimes V) \rightarrow M \otimes F_x(V)$  is defined on the components as

$$(F_x)_2^{a,M,V} : T^a(M \otimes V) \xrightarrow{(T^a)_2^{M,V}} T^a(M) \otimes T^a(V) \xrightarrow{\mu_M^a \otimes 1} M \otimes T^a(V).$$

On the other hand note that the  $\mathcal{C}^G$ -module functor structure of  $G_x \circ c_{H,x}$  on the components is given by:

$$\begin{aligned}
 (G_x c_{H,x})_2^{a,M,V} : T^{ax^{-1}}(T^x(M \otimes V)) &\xrightarrow{T^{ax^{-1}}(T^x)_2^{M,V}} T^{ax^{-1}}(T^x(M) \otimes T^x(V)) \rightarrow \\
 \xrightarrow{T^{ax^{-1}}(\mu_M^x \otimes 1)} T^{ax^{-1}}(M \otimes T^x(V)) &\xrightarrow{(T^{ax^{-1}})_2^{M,T^x(V)}} T^{ax^{-1}}(M) \otimes T^{ax^{-1}}(T^x(V)) \rightarrow \\
 \xrightarrow{\mu_M^{ax^{-1}} \otimes 1} M \otimes T^{ax^{-1}}(T^x(V)).
 \end{aligned}$$

Thus for any  $a \in \mathcal{R}_x$ , on the components, the above diagram is equivalent to the following:

$$\begin{array}{ccc}
 T^{ax^{-1}}(T^x(M \otimes V)) & \xrightarrow{(T_2^{a,x})_{M \otimes V}} & T^a(M \otimes V) \\
 \downarrow T^{ax^{-1}}((T_2^x)_{M,V}) & & \downarrow (T_2^a)^{M,V} \\
 T^{ax^{-1}}(T^x(M) \otimes T^x(V)) & \xrightarrow{\quad} T^{ax^{-1}}(T^x(M)) \otimes T^{ax^{-1}}(T^x(V)) \xrightarrow{\quad} T^a(M) \otimes T^a(V) & \\
 \downarrow T^{ax^{-1}}(\mu_M^x \otimes 1) & \searrow \text{dashed} & \downarrow \mu_M^a \otimes \text{id} \\
 T^{ax^{-1}}(M \otimes T^x(V)) & & T^a(M) \otimes T^{ax^{-1}}(T^x(V)) \\
 \downarrow (T^{ax^{-1},x})_2^{M,T^{ax^{-1}}(V)} \otimes \text{id} & \swarrow \text{dashed} & \downarrow \mu_M^a \otimes \text{id} \\
 T^{ax^{-1}}(M) \otimes T^{ax^{-1}}(T^x(V)) & & T^a(M) \otimes T^{ax^{-1}}(T^x(V)) \\
 \downarrow \mu_M^{ax^{-1}} \otimes \text{id} & \swarrow \text{dashed} & \downarrow \mu_M^a \otimes \text{id} \\
 M \otimes T^{ax^{-1}}(T^x(V)) & \xrightarrow{\text{id} \otimes (T_2^a)^{ax^{-1},x}_V} & M \otimes T^a(V)
 \end{array}$$

Note that the upper rectangle is commutative since  $T_2^{ax, x^{-1}}$  is a natural transformation of monoidal functors. The part below is commutative by the natural properties of the tensor bifunctor of the monoidal category  $\mathcal{C}$  and the compatibility between  $\mu_M^{ax^{-1}}$ ,  $\mu_M^{x^{-1}}$  and  $\mu_M^a$ .  $\square$

**Remark 4.19.** Note that  $\text{Rep}(G)$  can be regarded as the equivariantization  $\text{Vec}^G$  of the trivial action of  $G$  on  $\mathcal{C} = \text{Vec}$ , [5]. In this case the previous theorem recovers the usual Mackey decomposition for representations of finite groups.

**Lemma 4.20.** Suppose that a finite group  $G$  acts  $k$ -linearly on  $\mathcal{C}$ . Let  $K \leq H \leq L$  be a tower of subgroups of  $G$ . 1) One has that

$$\mathcal{R}_K^L = \mathcal{R}_K^H \mathcal{R}_H^L$$

2) There is a natural transformation

$$\mathbf{I}_{K,L}^H : \mathcal{I}_K^L \rightarrow \mathcal{I}_H^L \mathcal{I}_K^H$$

which is an isomorphism of  $k$ -linear functor.

3) Moreover if the action of  $G$  on  $\mathcal{C}$  is by monoidal equivalences then  $\mathbf{I}_{K,L}^H$  is an isomorphism of  $\mathcal{C}^L$ -module functors.

*Proof.* The natural transformation  $\mathbf{I}_{K,L}^H$  is defined as follows. Fix  $\mathcal{R}$  and  $\mathcal{S}$  sets of representative elements for the left cosets of  $K$  inside  $H$  and of  $H$  inside  $L$  respectively. Then  $\mathcal{RS} := \{rs \mid r \in \mathcal{R}, s \in \mathcal{S}\}$  is a set of representative for the left cosets of  $K$  inside  $L$ . Then for any  $M \in \mathcal{C}^K$  one has  $\mathcal{I}_L^K(M) = \bigoplus_{r \in \mathcal{R}} \bigoplus_{s \in \mathcal{S}} T^{rs}(M)$  while  $\mathcal{I}_H^K(M) = \bigoplus_{r \in \mathcal{R}} T^r(M)$  and  $\mathcal{I}_L^H(P) = \bigoplus_{s \in \mathcal{S}} T^s(P)$  for any  $P \in \mathcal{C}^H$ . Then one can define on the components

$$(\mathbf{I}_{K,L}^H)_M = \bigoplus_{r \in \mathcal{R}} \bigoplus_{s \in \mathcal{S}} (T_2^{r,s})_M.$$

Similarly to Lemma 4.8 one can check that  $(\mathbf{I}_{K,L}^H)_M$  is an isomorphism in  $\mathcal{C}^L$ . Thus one has to verify that the following diagram

$$\begin{array}{ccc} T^l(\text{Ind}_K^L(M)) & \xrightarrow{T^l((\mathbf{I}_{K,L}^H)_M)} & T^l(\text{Ind}_H^L \text{Ind}_K^H(M)) \\ \downarrow \mu_{\text{Ind}_K^L(M)}^l & & \downarrow \mu_{\text{Ind}_H^L \text{Ind}_K^H(M)}^l \\ \text{Ind}_K^L(M) & \xrightarrow{(\mathbf{I}_{K,L}^H)_M} & \text{Ind}_H^L \text{Ind}_K^H(M) \end{array}$$

is commutative for any  $l \in L$ . Indeed suppose that  $lsr = s'r'k'$  for some  $k \in K$  and  $s' \in \mathcal{S}$  and  $r' \in \mathcal{R}$ . Moreover suppose that  $ls = s''h$  and  $hr = r''k''$  for some  $r'' \in \mathcal{R}$ ,  $s'' \in \mathcal{S}$ ,  $h \in H$  and  $k'' \in K$ . Since  $lsr = s'r'k' = s''r''k''$  it follows that  $s' = s''$ ,  $r' = r''$  and  $k' = k''$ . Thus, on the components the above diagram is equivalent to the following diagram:

$$\begin{array}{ccc}
T^l(T^{sr}(M)) & \xrightarrow{T^l((T_2^{s,r})_M^{-1})} & T^l(T^s(T^r(M))) \\
\downarrow (T_2^{l,sr})_M & & \downarrow (T_2^a)^{M,V} \\
T^{lsr}(M) & \xrightarrow{(T_2^{ls,r})_M^{-1}} & T^{ls}(T^r(M)) \\
\downarrow (T_2^{s'r',k'})_M & \searrow & \downarrow (T_2^{s',h})_{T^r(M)}^{-1} \\
T^{s'r'}(T^{k'}(M)) & \xrightarrow{(T_2^{s',hr})_M^{-1}} & T^{s'}(T^h(T^r(M))) \\
\downarrow T^{s'r'}(\mu_M^{k'}) & \searrow & \downarrow T^{s'}((T_2^{h,r})_M) \\
T^{s'r'}(M) & \xrightarrow{(T_2^{s',r'})_{T^k(M)}^{-1}} & T^{s'}(T^{hr}(M)) \\
& \searrow & \downarrow T^{s'}((T_2^{r',k'})_M^{-1}) \\
& & T^{s'}(T^{r'}(T^{k'}(M))) \\
& & \downarrow T^{s'}(T^{r'}(\mu_M^{k'})) \\
& & T^{s'}(T^{r'}(M))
\end{array}$$

2) Suppose now that  $\mathcal{C}$  is a  $k$ -linear monoidal category and the action of  $G$  is by monoidal autoequivalences. Let  $M \in \mathcal{C}^L$  and  $r \in \mathcal{R}$ ,  $s \in \mathcal{S}$ . The fact that  $\mathcal{I}_H^K$  is natural transformation of  $\mathcal{C}^L$ -module functors follows from the commutativity of the diagram below.

$$\begin{array}{ccccc}
T^{rs}(M \otimes V) & \xrightarrow{(T_2^{r,s})_{M \otimes V}} & T^r(T^s(M \otimes V)) & & \\
\downarrow (T_2^{rs})^{M,V} & & \downarrow T^r((T_2^s)^{M,V}) & & \\
T^{rs}(M) \otimes T^{rs}(V) & \xrightarrow{(T_2^{r,s})_M^{-1} \otimes (T_2^{r,s})_V^{-1}} & T^r(T^s(M)) \otimes T^r(T^s(V)) & \xrightarrow{(T_2^r)^{T^s(M), T^s(V)}} & T^r(\mu_M^s \otimes 1) \\
\downarrow \mu_M^{rs} \otimes 1 & & \downarrow T^r(\mu_M^s) \otimes 1 & & \downarrow T^r(\mu_M^s \otimes 1) \\
M \otimes T^{rs}(V) & \xleftarrow{\mu_M^r \otimes (T_2^{r,s})_V} & T^r(M) \otimes T^r(T^s(V)) & \xleftarrow{(T_2^r)^{M, T^s(V)}} & T^r(M \otimes T^s(V))
\end{array}$$

The upper pentagon is commutative by Equation (2.14). The bottom left square is commutative by Equation (2.8). The bottom right square is commutative by the naturality of the transformation  $T_2^r$  with respect to the first argument.  $\square$

**Proposition 4.21.** *Suppose that  $\mathcal{C}, \mathcal{D}$  and  $\mathcal{E}$  are rigid monoidal categories and  $F_1 : \mathcal{C} \rightarrow \mathcal{D}$  and  $F_2 : \mathcal{C} \rightarrow \mathcal{E}$  are two monoidal functors with left adjoint functors  $I_1 : \mathcal{D} \rightarrow \mathcal{C}$  and respectively  $I_2 : \mathcal{E} \rightarrow \mathcal{C}$ . Then for any objects  $M \in \mathcal{O}(\mathcal{D})$  and  $N \in \mathcal{O}(\mathcal{E})$  one has the canonical isomorphism in  $\mathcal{C}$*

$$(4.22) \quad I_1(M) \otimes I_2(N) \simeq I_1(F_1(I_2(N)) \otimes M) \simeq I_2(F_2(I_1(M)) \otimes N).$$

*Proof.* It can be shown by a straightforward computation that

$$(4.23) \quad \text{Hom}_{\mathcal{C}}(I_1(M) \otimes I_2(N), P) \simeq \text{Hom}_{\mathcal{C}}(I_2(F_2(I_1(M)) \otimes N), P)$$

for any object  $P \in \mathcal{C}$ . Indeed,

$$\begin{aligned} \text{Hom}_{\mathcal{C}}(I_1(M) \otimes I_2(N), P) &= \text{Hom}_{\mathcal{C}}(I_1(M), P \otimes I_2(N)^*) = \\ &= \text{Hom}_{\mathcal{D}}(M, F_1(I_2(N)^* \otimes P)) = \text{Hom}_{\mathcal{D}}(M, F_1(I_2(N))^* \otimes F_1(P)) = \\ &= \text{Hom}_{\mathcal{D}}(F_1(I_2(N)) \otimes M, F(P)) = \text{Hom}_{\mathcal{C}}(I_1(F_1(I_2(N)) \otimes M), P) \end{aligned}$$

Then Yoneda's lemma implies the conclusion.  $\square$

In particular for  $\mathcal{E} = \mathcal{C}$  and  $F_2 = I_2 = \text{id}_{\mathcal{C}}$  one obtains that

$$(4.24) \quad I_1(M) \otimes V \simeq I_1(M \otimes F_1(V))$$

for any objects  $M \in \mathcal{D}$  and  $V \in \mathcal{C}$ .

**4.3. Proof of Theorem 1.1.** 1) First we will define the functors  $\mathcal{R}_H^L, \mathcal{I}_H^L$  and  $c_{H,a}$ .

The functors  $\mathcal{R}_L^H$  correspond to the restriction functors defined in section 2.3. We fix an arbitrary set of representative elements for the left cosets  $Hx$  of any group inclusion  $H \leq L$ . We consider the corresponding induction functor  $\text{Ind}_H^L$  as in Equation (2.11). Moreover, the conjugation functors  $c_{H,a}$  are defined as in Lemma 4.1.

Secondly we will define the natural transformation from the definition of the categorical Mackey functor. Note that the natural transformation  $\mathbf{R}_{K,L}^H$  can be taken the identity by the first item of Lemma 4.20. Also one can take  $\mathbf{CR}_{H,a}^K$  as identity functors by Equation (4.9) of Lemma 4.8. By Proposition 4.4 one can define  $(\mathbf{C}_{a,b}^H)_M := (T_2^{a,b})_M^{-1}$  for any object  $M \in \mathcal{C}^{abH}$ . Moreover the natural transformation  $\mathbf{I}_{K,L}^H : \mathcal{I}_K^L \rightarrow \mathcal{I}_H^L \mathcal{I}_K^H$  is defined by Lemma 4.20. The definition of the natural transformations  $\mathbf{CI}_{H,a}^K$  is given in Lemma 4.8.

Now one has to verify all the compatibility conditions from the definition of a categorical Mackey functor. The Mackey decomposition from Equation (3.2) is proven in Theorem 1.2. Clearly the identities (3.3), (3.4) and (3.5) hold. Moreover the diagrams (R), (RCC), (RRC) are commutative since all the natural transformations are the identity functors. Diagram (C) is commutative by Equation (2.4). The commutativity of the diagrams (ICC), (IIC), (I) follow by straightforward computation taking care of the representative elements for the left cosets of any inclusion of subgroups of  $G$ .

2) Suppose now that the action of  $G$  is by monoidal autoequivalences on the  $k$ -linear monoidal category  $\mathcal{C}$ . Note that we have already noticed that the restriction functors  $\text{Res}_L^H$  are monoidal functors. By the second part of Lemma 4.1 one has that  $c_{H,a}$  are also monoidal functors and the natural transformations  $\mathbf{C}_{a,b}^H$  are morphisms of monoidal functors. The natural transformations  $\mathbf{R}_{K,L}^H$  and  $\mathbf{CR}_{H,a}^K$  are automatically morphisms of module functors since they are the identity functors. Note that the condition (CG6) follows from Proposition 4.21. All the other additional compatibility structures required for a categorical Green functor follow from the above lemmata.  $\square$



**Theorem 4.25.** *Let  $G$  be a group and  $\mathcal{M}$  be a categorical  $G$ -Mackey functor over the monoidal category  $\mathcal{S}$ . With the above notations one has the following:*

- 1) *Any categorical Mackey functor  $\mathcal{M}$  gives a usual Mackey functor  $L \mapsto K_i(\mathcal{M}(L))$  by taking the  $K$ -theory over  $K_0(\mathcal{S})$ .*
- 2) *Moreover, if  $\mathcal{M}$  is a categorical Green functor then  $K_0$  gives a Green functor  $L \mapsto K_0(\mathcal{M}(L))$  over  $K_0(\mathcal{S})$ .*

*Proof.* Similarly to [7] one can use the following elementary facts about  $K$ -theory (see [10]): If  $F_1$  and  $F_2$  are isomorphic exact functors on an exact category, then they induce the same map on  $K$ -theory; and if  $F_1$  and  $F_2$  are exact functors on an exact category inducing homomorphisms  $f_1$  and  $f_2$  on  $K$ -groups, then the functor  $F_1 \oplus F_2$  induces the homomorphism  $f_1 + f_2$ . Now identities (M1) - (M4) from the definition of a Mackey functor follow from their functorial counterparts given in the definition of the categorical Mackey functor. Moreover, if  $F$  is a monoidal functor then it induces an algebra morphism at the level of the Grothendieck rings. The adjunction properties from the definition of a Green functor follow from the Proposition 4.21.  $\square$

Applying Theorem 4.25 to the Mackey functor from Theorem 1.1 we obtain Corollary 1.4. We finish the paper with the following two examples previously considered in the literature.

**Example 4.26.** *Suppose that  $R \subset S$  is a Galois extension of rings with Galois Group  $G$ . Then as in Example 2.10 the group  $G$  acts on the category  $S\text{-mod}$  and  $(S\text{-mod})^G \simeq S\#\mathbb{Z}G\text{-mod}$ . Since  $S^G$  is Morita equivalent to  $S\#\mathbb{Z}G$  and the  $K$ -theory is preserved by Morita equivalence it follows by Corollary 1.4 that  $H \mapsto K_i(S^H)$  is a Mackey functor. Thus our results extends the results obtained in [7] for Galois extensions of commutative rings.*

**Example 4.27.** *Suppose that we have a cocentral extension of semisimple Hopf algebras*

$$(4.28) \quad k \rightarrow B \xrightarrow{i} H \xrightarrow{\pi} kF \rightarrow k.$$

*Recall that this means the above sequence is exact, (see [13]), and that  $kF^* \subset \mathcal{Z}(H^*)$  via  $\pi^*$ . Following [9, Proposition 3.5] it follows that  $F$  acts on the fusion category  $\text{Rep}(B)$  and  $\text{Rep}(H) = \text{Rep}(B)^F$ . Recall from [9] that for all  $x \in F$  the action is given by  $T^x(M) = M$  as vector spaces with the action of  $B$  on  $T^x(M)$  given by  $b \cdot {}^x m = (x^{-1} \cdot b)m$ . Note that there is a typo in defining this action in [9, Section 3.2]. With this categorical group action the Green functors from Corollary 1.4 coincide to those described in [2, Theorem 5.8].*

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